MAGNETOHYDRODYNAMICS   MAGNETOGASDYNAMICS

DTIC
ELECTE
MAY 12 1994
S
F
D

94-14045

94 5 09 135

BIBLIOGRAPHICAL CONTROL SHEET

1. (Originating agency and/or monitoring agency)

   O.A.: University of Maryland, College Park, Maryland

   M.A.: ARDC Air Force Office of Scientific Research

2. (Originating agency and/or monitoring agency report number)

   O.A.: BN-59

   M.A.: OSR-TN-55-???

3. (Title and classification of title)

   Magnetofluiddynamics (Unclassified)

4. Personal author: S. I. Pai

5. Date of report: September 1958

6. Pages: 35

7. Illustrative material: None

8. Prepared on Air Force contract: AF18(603)???

9. Prepared for Project Code: R-352-30-17

10. **Security classification:** Unclassified

11. **Distribution limitations:** None

12. **Abstract:** The fundamental equations of magnetofluiddynamics are derived. For incompressible fluid, magnetohydrodynamics, the important parameters are the Reynolds number R_e, the magnetic pressure number R_H, which is the ratio of magnetic pressure to dynamic pressure, and the velocity number R_v, which is the ratio of the fluid velocity to the characteristic velocity of which the magnetic field is moving through a conductor. Some exact solutions and properties of the equations of magnetohydrodynamics are given. Stability of laminar flow and turbulence in magnetohydrodynamics are briefly reviewed. Finally some magnetohydrodynamic experiments are described.

For compressible fluid, magnetogasdynamics, the important parameters are still R_e, R_H and R_v plus other well known important parameters of ordinary gas dynamics such as Mach number, Prandtl number and ratio of specific heats. Both the waves of small amplitude, Alfven's waves in compressible fluid, and shock waves in magnetogasdynamics are discussed.

1. Introduction.

Electromagnetic phenomena in solid conductors have been well known for a long time but the electromagnetic phenomena in fluids, liquid or gases, are not so well known. Only recently the problem of magnetohydrodynamics has been attracting the attention of some research workers because it is important in astrophysics, geophysics and the behavior of interstellar gas masses.[1-4] The main difference in the electromagnetic phenomena in fluids from those in solids is due to the fact that mechanical forces deriving from electric currents may produce fluid dynamic motions and the fluid dynamic motions may produce electromagnetic phenomena. In other words, there is an interaction between the electromagnetic forces and the ordinary fluid dynamic forces. This interaction phenomena becomes important whenever the electromagnetic forces are of the same order of magnitude as the inertial forces or viscous forces of the fluid. Actually the discovery of magnetohydrodynamics was made in cosmic physics. Magnetic phenomena of different types has been observed and needed an explanation. In ordinary laboratory experiments of discharges of electrical current in gases, the mechanical effects are usually small, the results of these experiments cannot be used to explain the electromagnetic phenomena affecting the motion of gaseous masses of cosmical dimensions. Hence study of electromagnetic phenomena in liquid conductors has been extensively carried out where in some respects

the conditions are more similar to those occurring in cosmical

physics. Thus the magnetohydrodynamics has been found. The

result of magnetohydrodynamics has also some engineering

interest because of the utility of induction flow-meters, which

rely on the generation of a measurable potential difference in

the fluid in a direction perpendicular to the motion and to the

magnetic field.

Recently because the interest of hypersonic flow of missiles,

the interaction of the electromagnetic force and fluid dynamic

forces in gases are not negligible and may be produced in lab-

oratories by the use of shock tube. Thus some preliminary

studies of magnetogasdynamics has been initiated. But much

has to be done before a complete understanding of these phen-

omena is possible. In this present paper, I shall make a brief

review of the present status of magnetohydrodynamics and magneto-

gasdynamics.

2. Fundamental Equations.

In studying the magnetic fluid dynamics, one has to deal

with the fluid dynamical equations and the electrodynamic equa-

tions simultaneously. In the general three dimensional flow

there are 16 unknowns in the problems of a magnetofluiddynamics,

i.e.;

(a) The magnetic field strength $\vec{H}$ (3 components)

(b) The electric field strength $\vec{E}$ (3 components)

(c) The electric current density $\vec{J}$ (3 components)

(d) The excess electric charge ρ_e

(e) The fluid velocity vector $\vec{q}$ (3 components)

(f) The pressure of the fluid p

(g) The density of the fluid ρ

(h) The temperature of the fluid T.

For these unknowns, we have to find 16 relations which are the fundamental equations of magnetofluiddynamics and which are given below:

The Maxwell equations[5] are assumed to be true in magneto-fluiddynamics. They are six in number and are as follows:

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \epsilon \vec{E}}{\partial t} \tag{2.1}$$

$$\nabla \times \vec{E} = - \frac{\partial \mu_e \vec{H}}{\partial t} \tag{2.2}$$

where ∇ is the gradient operator. Gibbs vector notations are used. t is the time. ϵ is the dielectric constant and μ_e is the magnetic permeability. The MKS unit system is used.

The current density equation is

$$\vec{J} = \sigma\left(\vec{E} + \mu_e \vec{q} \times \vec{H}\right) + \rho_e \vec{q} \tag{2.3}$$

where σ is the electric conductivity. The terms with $\vec{q}$ are the coupling terms with the fluid dynamic equations which represent the interaction phenomena.

The conservation of electric charge gives

$$\nabla \cdot \vec{J} + \frac{\partial \rho_e}{\partial t} = 0 \tag{2.4}$$

Equations (2.1) to (2.4) are the electromagnetic equations with coupling terms due to fluid dynamic motion.

The conservation of mass, i.e. the equation of continuity in fluid dynamics, is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{q}) = 0 \qquad (2.5)$$

For incompressible fluid, ρ = constant = ρ_0, equation (2.5) becomes

$$\nabla \cdot \vec{q} = 0 \qquad (2.5a)$$

The equation of motion is as follows

$$\rho \left[\frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla) \vec{q} \right] = -\nabla p + \rho_e \vec{E} + \mu_e (\vec{J} \times \vec{H}) + \nabla \cdot \underline{\tau} \qquad (2.6)$$

where $\underline{\tau}$ is the stress tensor due to viscosity which is usually assumed to be linear homogeneous function of rate of change of velocity. The i j component of the stress tensor τ_{ij} may be written as follows:

$$\tau_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3} \mu (\nabla \cdot \vec{q}) \delta_{ij} \qquad (2.7)$$

where $i, j = 1, 2, 3$. Subscript i or j refers to the i or j component of the vector respectively. u_i is the ith component of velocity vector q and x_i is the ith component of cartesian spatial coordinate. $\delta_{ij} = 0$ if $i \neq j$, $\delta_{ij} = 1$ if $i = j$. μ is the ordinary coefficient of viscosity. We assume that the second coefficient of viscosity is $-\frac{2}{3} \mu$. For incompressible fluid with constant viscosity, the viscosity terms may be greatly simplified and equation (2.6) becomes

$$\rho\left[\frac{\partial \vec{q}}{\partial t} + (\vec{q}\cdot\nabla)\vec{q}\right] = -\nabla p + \left(\rho_e \vec{E} + \mu_e \vec{J}\times\vec{H}\right) + \mu \nabla^2\vec{q} \qquad (2.6a)$$

where $\nabla^2 = \nabla\cdot\nabla$.

It is an empirical fact that there is a functional relation between the density ρ , the pressure p and the temperature T of a fluid. This relation is known as equation of state. For gas dynamics, the perfect gas law is usually used which is

$$p = \rho R T \qquad (2.8)$$

where R is the gas constant. In magnetogasdynamics equation (2.8) is also used as the equation of state.

In magnetohydrodynamics, equation (2.8) is replaced by

$$\rho = \rho_o = \text{constant} \qquad (2.8a)$$

which is assumed to be known.

The last relation is the energy equation which is

$$\frac{\partial E_t}{\partial t} + u_i\frac{\partial E_t}{\partial x_i} + \left(E_t + p\right)\frac{\partial u_i}{\partial x_i} + \frac{\partial Q}{\partial t} + \left(\tau_{ij}^m + \tau_{ij}^E\right)\frac{\partial u_i}{\partial x_j}$$
$$+ \frac{\partial}{\partial x_j}\left(K\frac{\partial T}{\partial x_j}\right) - \frac{\partial S_i}{\partial x_i} - E_i J_i - \frac{\partial q_{Ri}}{\partial x_i} = 0 \qquad (2.9)$$

where the summation convention is used, i.e.

$$\frac{\partial u_i}{\partial x_i} = \frac{\partial u_i}{\partial x_i} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \qquad \text{and}$$

$E_t = \rho I + U_E + U_H$

I = internal energy of the fluid per unit mass

$= C_v T$ for perfect gas, C_v = specific heat at constant volume.

$U_E = \frac{\epsilon}{2} E_i E_i$ = electric energy per unit volume.

$U_H = \frac{\mu_e}{2} H_i H_i = $ magnetic energy per unit volume.

$\frac{\partial Q}{\partial t} = $ rate of energy produced by external agencies. For adiabatic system $\frac{\partial Q}{\partial t} = 0$.

$\tau_{ij}^M = $ ij component of viscous stress tensor given by (2.7).

$\tau_{ij}^E = $ ij component of electromagnetic stress tensor

$$= \epsilon E_i E_j + \mu_e H_i H_j - \frac{1}{2}\left[\epsilon E_k E_k + \mu_e H_k H_k\right]\delta_{ij}$$

$K = $ coefficient of heat conductivity

$S_i = $ ith component of the Poynting vector $\vec{S}$, where

$$\vec{S} = \vec{E} \times \vec{H}$$

$q_{Ri} = $ ith component of the radiation energy flux.

$$\frac{\partial q_{Ri}}{\partial x_i} = R_E - R_A$$

$R_E = $ rate of radiation energy emission per unit volume.

$R_A = $ rate of radiation energy absorption per unit volume.

In the case of incompressible fluid, i.e. magnetohydrodynamics, if we assume that the coefficient of viscosity μ is constant, we may solve the unknowns $\vec{H}, \vec{E}, \vec{J}, \rho_e, \vec{q}$ and p from equation (2.1) to (2.6) without considering the energy equation. After these unknowns are found, equation (2.9) gives the temperature. For compressible fluids, i.e. magnetogasdynamics, we have to deal with equations (2.1) to (2.9) simultaneously. In the following we shall review briefly the present status of magnetohydrodynamics and magnetogasdynamics.

Part L. MAGNETO-HYDRODYNAMICS

3. Important Parameters in Magnetohydrodynamics.

We consider only the cases where the velocity of the flow $\vec{q}$ is much smaller than the velocity of light C. The relativistic effects may be omitted. Since the energy in the electric field is of the order of q^2/C^2 of the energy in the magnetic field, in this case, the energy in the electric field can be neglected. Consequently in ordinary magnetohydrodynamics, we consider only the interaction between the velocity field $\vec{q}$, and the magnetic field $\vec{H}$. In this approximation we may put $\rho_e = 0$ in the fundamental equations.

The fundamental equations for magnetohydrodynamics may be obtained from equations (2.1) to (2.6) in the following non-dimensional forms:

$$\frac{\partial \vec{q}^*}{\partial t^*} + \left(\vec{q}^* \cdot \nabla^*\right)\vec{q}^* - R_H\left(\vec{H}^* \nabla^*\right)\vec{H}^* = -\nabla^*\left(p^* + \frac{R_H}{2}\vec{H}^{*2}\right) + \frac{1}{R_e}\nabla^{*2}\vec{q}^* \tag{3.1}$$

$$\frac{\partial \vec{H}^*}{\partial t^*} + \left(\vec{q}^* \cdot \nabla^*\right)\vec{H}^* - \left(\vec{H}^* \cdot \nabla^*\right)\vec{q}^* = \frac{1}{R_V}\nabla^{*2}\vec{H}^* \tag{3.2}$$

$$\nabla^* \cdot \vec{q}^* = 0 \tag{3.3}$$

where the non-dimensional quantities are defined as follows:

$$\vec{q}^* = \frac{\vec{q}}{U} \quad , \quad \vec{H}^* = \frac{\vec{H}}{H_o} \quad , \quad \nabla^* = L\nabla \quad , \quad t^* = \frac{U}{L}t$$

$$p^* = \frac{p}{\rho_o U^2} \quad , \quad R_H = \frac{\mu_e H_o^2}{\rho_o U^2} \quad , \quad R_e = \frac{\rho_o UL}{\mu} \quad , \quad R_V = \sigma\mu_e UL \tag{3.4}$$

U is the characteristic velocity, L is the characteristic
length and H_0 is the characteristic magnetic field of this
system. The problem is to solve the unknowns p^*, $\vec{q}^*$ and $\vec{H}^*$
in terms of the important parameters R_e, R_H and R_v.

The meanings of these three important parameters are as
follows:

(1) R_e is the well known Reynolds number which character-
izes the viscous flow.

$$R_e = \frac{\text{Inertial force}}{\text{Viscous force}} = \frac{\rho_o\, UL}{\mu} \qquad (3.5)$$

(2) R_H is the magnetic pressure number which is the ratio
of magnetic pressure $\frac{\mu_e\, H_o^2}{2} = \frac{\rho_o}{2}\, V_H^2$, say, over the dynamic
pressure $\rho_o\, U^2/2$.

$$R_H = \frac{\text{magnetic pressure}}{\text{dynamic pressure}} = \frac{\mu_e\, H_o^2/2}{\rho_o\, U^2/2} = \frac{V_H^2}{U^2} \qquad (3.6)$$

Only when R_H is of the order of unity or larger the fluid flow
will be affected by the magnetic field. If $R_H \ll 1$, the terms
due to magnetic field in the equations of motion may be neglected,
and the $\vec{q}$ will not be affected noticeably by $\vec{H}_o$

(3) R_v is the velocity number which is

$$R_v = \sigma \mu_e L\, U = \frac{U}{V_e} \qquad (3.7)$$

where $V_e = \frac{1}{\sigma \mu_e L}$. "V_e" may be regarded as a characteristic
velocity of which the magnetic field is moving through the con-
ductor. If $U \gg V_e$, the field is practically compelled to
follow the motion, the magnetic field will be greatly influenced

by the motion. On the other hand, if $U \ll V_e$, the magnetic field will not be influenced noticeably by the motion. This is the reason why the magnetohydrodynamic dimension L is very large so that V_e is very small.

4. **Solutions and Properties of the Equations of Magnetohydrodynamics.**

It is well known that there is no general method of solution for hydrodynamical equations alone. Thus it is much more difficult to investigate magnetohydrodynamics. One of the first steps to understand magnetohydrodynamic equations is to find some simple exact solutions of magnetohydrodynamical equations in order to bring out its essential features. One way to find such exact solutions will be the generalization of some well known results of ordinary hydrodynamics. The following gives some of these results:

(a) Two dimensional steady flow between parallel plates.

Consider two parallel plates situated in the plane $x_2^* = \pm 1$ respectively, x_2^* being one of the non-dimensional spatial coordinates x_i^*. Let only one component (x_1^* - wise) of velocity be different from zero and be a function of x_2^* only, i.e. $u_1^*(x_2^*)$. Furthermore we assume that there is a uniform external magnetic field H_0 which is in the x_2^* - direction. It should be noticed that if H_0 is in the x_1^* or x_3^* - direction, the velocity field will not be influenced by the magnetic field in this case. Under these conditions, the

equations (3.1) to (3.3) give

$$\frac{d^2 u_1^*}{d x_2^*} - k^2 \frac{d u_1^*}{d x_2^*} = 0 \qquad (4.1)$$

where $\quad k^2 = R_H R_e R_v = \frac{H_0^2 \mu_e^2 L^2 \sigma}{\mu} \qquad (4.2)$

The general solution of equation (4.1) is

$$u_1^* = \frac{A}{k} \cosh k x_2^* + \frac{B}{k} \sinh k x_2^* + C \qquad (4.3)$$

where A, B and C are arbitrary constants to be determined from the boundary conditions.

(1) Plane Couette flow. The boundary conditions are $x_2^* = 0$, $u_1^* = 0$; $x_2^* = +1$, $u_1^* = +1$, $x_2^* = -1$, $u_1^* = -1$. Equation (4.3) becomes

$$u_1^* = \frac{\sinh k \, x_2^*}{\sinh k} \qquad (4.4)$$

This flow corresponds to the flow between two parallel infinite plates moving in opposite direction with velocity $\pm U$ and at the location $x_2^* = \pm 1$ respectively. There is no pressure gradient in x_1^* - direction.

If "k" tends to be zero, equation (4.4) becomes

$$u_1^* = x_2^* \qquad (4.5)$$

which is the linear velocity distribution of ordinary hydrodynamics.

If "k" is very large, $k \rightarrow \infty$, then

$$\lim_{k \to \infty} u_1^* = \lim_{k \to \infty} \frac{\sinh k x_2^*}{\sinh k} \longrightarrow 0 \qquad (4.6)$$

except $x_2^* \longrightarrow \pm 1$. For very large value of k, in the central portion between the two plates, the velocity is almost zero and then increases very rapidly to the values ± 1 near $x_2^* = \pm 1$. This is a typical phenomenon in magnetohydrodynamics of large k. The reason is due to the fact that the magnetic field increases the total shearing stress of the flow field. Near the wall, the shearing stress due to the magnetic field tends to zero, the total shearing stress will be largely produced by the viscous force only. As a result there will be a large velocity gradient near the wall. This effect is qualitatively similar to that of turbulent flow. In turbulent flow the total shearing stress of the flow is much larger than that of the corresponding laminar flow. But the turbulent stress will be zero at the wall; thus there will be a large velocity gradient near the wall of turbulent flow to respond to the large shearing stress of the flow field.

(ii) Plane Poiseuille Flow, [6,7] The boundary conditions are $x_2^* = \pm 1$, $u_1^* = 0$; $x_2^* = 0$, $u_1^* = 1$. Equation (4.3) becomes

$$u_1^* = \frac{\cosh k - \cosh k x_2^*}{\cosh k - 1} \qquad (4.7)$$

If $k \to 0$, (4.7) becomes

$$u_1^* = 1 - x_2^{*2} \qquad (4.8)$$

This is the plane Poiseuille flow of ordinary hydrodynamics. If $k \to \infty$,

$$\lim_{k \to \infty} \frac{\cosh k x_2^*}{\cosh k} \longrightarrow 0$$

except $x_2^* \to \pm 1$, hence

$$u_1^* \cong 1 \tag{4.9}$$

except $x_2^* = 1$. Here again we have a large velocity gradient near the walls.

After the value of u_1^* is obtained, the induced magnetic field in x_1^* - direction H_{x1}^* and the pressure p^* may be obtained by quadrature.

$$H_{x_1}^* = R_V \left[\frac{K_0}{2} (x_2^* + 1) - \int_{-1}^{x_2^*} u_1^* d x_2^* \right] \tag{4.10}$$

where the boundary conditions $x_2^* = \pm 1$, $H_{x1}^* = 0$ are used and

$$K_0 = \int_{-1}^{1} u_1^* d x_2^*$$

$$p^* = - R_H \frac{H_{x_1}^{*2}}{2} + A_{p_1} X_1^* + B_p \tag{4.11}$$

where A_p and B_p are constants. A_p is the pressure gradient in x_1^* - direction, (dp^*/dx_1^*). B_p is the pressure at $x_1^* = 0$, $x_2^* = \pm 1$. It is interesting to notice that x_1^* wise pressure variation is the same here as that in the ordinary hydrodynamics. The magnetic field introduces the variation of pressure in the x_2^* - direction. This point again is similar to the case of turbulent flow of ordinary hydrodynamics.

(b) Other simple steady flows. There are many other simple steady flow exact solutions had been found, e.g.:

(i) Flow in a circular pipe under an external radial magnetic field. (ref. 8)

(ii) Flow in a rectangular channel under transverse magnetic fields (ref. 9).

(iii) Flow between concentric circular cylinders under constant axial magnetic field (ref. 10). In this case, the velocity distribution is not affected by the presence of the magnetic field.

(c) Analogy between magnetic field and vorticity.

There is a formal analogy between the vorticity $\vec{\omega}$ of ordinary hydrodynamics and the magnetic field $\vec{H}$ of magnetohydrodynamics[16]. The fundamental equations for vorticity $\vec{\omega}$ are

$$\nabla \cdot \vec{\omega} = 0$$

$$\frac{\partial \vec{\omega}}{\partial t} - \nabla \times (\vec{q} \times \vec{\omega}) = \omega \, \nabla^2 \vec{\omega} \qquad (4.12)$$

The fundamental equations for the magnetic field H are

$$\nabla \cdot \vec{H} = 0$$

$$\frac{\partial \vec{H}}{\partial t} - \nabla \times (\vec{q} \times \vec{H}) = \omega_H \, \nabla^2 \vec{H} \qquad (4.13)$$

where $\omega_H = \frac{1}{\mu_e \sigma} \doteq$ magnetic diffusivity.

Since equations (4.12) and (4.13) are identical in form, we may apply all of the known theorems of vorticity in ordinary hydrodynamics to the magnetic field of magnetohydrodynamics. For instance, from Helmholtz's theorem, we may show that when conductivity σ is infinite, i.e. $\omega_H = 0$, the lines of magnetic force move with the fluid.

5. Stability of Laminar Flow in Magnetohydrodynamics.

It has now been generally recognized that turbulent motion is the more natural state of fluid flow, and the laminar motion occurs only the Reynolds number is so low that the deviations from it are liable to be damped out. In cosmic conditions, where the problem of magnetohydrodynamics is important, because of large dimensions, i.e. large Reynolds numbers, one would expect that the flow will be mostly turbulent. There are two problems of great interest in these conditions. One is the problem of stability of laminar motion with respect to infinitesimally small disturbances. It should be noted that instability does not necessarily lead to turbulent motion, it could lead to another state of laminar motion. Another problem is, of course, the turbulent flow itself.

The ordinary theory of stability of laminar flow has been extended to include the effects of magnetic forces. All of these investigations consider those cases where the basic flow is not affected by the magnetic forces but the disturbances[10-14]. In all these cases, it was found that the magnetic field tends to increase the stability of laminar flow because the disturbances tend to be damped out by the eddy currents.

It is well known that the stability of the flow is greatly affected by the profile of the basic flow. It will be of interest if the effect of stability of laminar flow by the magnetic field should be investigated when both the effects of induced magnetic force and of the change of basic velocity profile occur, e.g.,

the problem of parallel flow with a transverse magnetic field discussed in § 4.

Another interesting point is that the Squire's theorem[15] for the relation of three dimensional and two dimensional disturbances in the stability problem of parallel flow, does not in general, hold in magnetohydrodynamics.

6. Turbulence in Magnetohydrodynamics.

The interaction between the electromagnetic and the hydrodynamic forces in an incompressible conducting fluid in turbulent motion was first studied systematically by Batchelor[16] and was further developed by Chandrasekhar[17] and others. The fundamental equations are still equations (3.1) to (3.3). We review some of the results of turbulent flow as follows:

(a) Growth[16] of electromagnetic energy in a turbulent motion. Batchelor considered the disturbance of electromagnetic energy produced by the turbulent motion in the absence of external electrical or magnetic field. He assumed that both the turbulent velocity and the electromagnetic fields are stationary random functions. From equation (4.13), it may be shown that the rate of change of the average amount of magnetic energy in unit volume of fluid is

$$\frac{1}{2} \overline{\frac{d |\vec{H}^2|}{dt}} = \overline{|\vec{H}^2| \left(\frac{\partial \vec{g}}{\partial x_H}\right)_H} - \omega_H \overline{|\nabla H_i|^2} \tag{6.1}$$

where the subscript H refers to the component in the direction of the lines of magnetic force and the repeated suffix i indicates

summation over three orthogonal components.

In the absence of electromagnetic effects, for isotropic turbulence, it is well known that the mean square vorticity satisfies the equation

$$\frac{1}{2} \frac{d\overline{|\vec{\omega}^2|}}{dt} = \overline{|\vec{\omega}^2| \left(\frac{\partial \vec{\xi}}{\partial x_{ii}}\right)_{ij}} - \vartheta \overline{|\nabla \omega_i|^2} \tag{6.2}$$

Here the analogy of magnetic field and vorticity may be applied by comparing equations (6.1) and (6.2). From experiments of ordinary isotropic turbulence, it was found that the rate of change of $|\overline{\vec{\omega}^2}|$ is approximately zero, i.e. the production term $\overline{|\vec{\omega}^2| \left(\frac{\partial \vec{\xi}}{\partial x_\omega}\right)_\omega}$ and the decay term $\omega \overline{|\nabla \omega_i|^2}$ are approximately in equilibrium. If one assumes that the statistical distributions $\vec{H}$ are approximately the same as those of $\vec{\omega}$, the two contributions to $d|\overline{H^2}|/dt$ will be in equilibrium if $\omega_H = \omega$; if $\omega_H > \omega$, $|\overline{H^2}|$ will decay to zero and if $\omega_H < \omega$, $|\overline{H^2}|$ will increase. Hence the criterion for the energy of the disturbing magnetic field to increase is

$$\omega \mu_e \sigma > 1 \tag{6.3}$$

(b) The invariant theory of isotropic turbulence in magnetohydrodynamics[17].

Chandrasekhar extended the statistical theory of isotropic turublence of ordinary hydrodynamics developed by Taylor, von Karman and others to the case with the presence of electromagnetic field. It is convenient to write the fundamental equations (3.1) to (3.3) in the following form

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_k}\left(u_i u_k - h_i h_k\right) = -\frac{1}{\rho}\frac{\partial}{\partial x_i}\left(p + \frac{1}{2}\rho h_k h_k\right) + \nu \nabla^2 u_i$$

$$\frac{\partial h_i}{\partial t} + \frac{\partial}{\partial x_k}\left(h_i u_k - h_k u_i\right) = \nu_H \nabla^2 h_i \tag{6.4}$$

where $h_i = \left(\frac{\mu_e}{\rho}\right)^{\frac{1}{2}} H_i$. h_i has the dimensions of a velocity.
Subscripts i and k refer to the ith and kth component of the
vector respectively and the summation convention is used. In
the derivation of (6.4) the solenoidal properties of the vectors
$\vec{q}$ and $\vec{H}$ are used.

In the statistical theory of homogeneous and isotropic
turbulence of Taylor and von Karman, we consider the correlation
between the fluctuating quantities at two points $(P(x_i)$ and
$P'(x_i')$ in space. From equation (6.4), we see that the follow-
ing correlations should be considered:

Double correlations:

$$\overline{u_i u_j'} \quad , \quad \overline{h_i h_j'} \quad \text{and} \quad \overline{u_i h_j'} \tag{6.5}$$

Triple correlations:

$$\overline{u_i u_j u_k'} \, , \, \overline{h_i h_j u_k'} \, , \, \overline{u_i u_j h_k'} \, , \, \overline{h_i h_j u_k'}$$

$$\overline{\left(h_i u_j - u_i h_j\right) h_k'} \quad \text{and} \quad \overline{u_i\left(h_j' u_k' - h_k' u_j'\right)} \tag{6.6}$$

where the primes and the lack of primes refer to the quantities
at the point P' and P respectively.

For homogeneous and isotropic turbulence, we may apply the theory of invariants to the tensors of (6.5) and (6.6). For double correlations, we have

$$\overline{u_i u_j'} = \frac{Q}{r} \xi_i \xi_j - (rQ' + 2Q)\delta_{ij}$$

$$\overline{h_i h_j'} = \frac{H'}{r} \xi_i \xi_j - (rH' + 2H)\delta_{ij}$$

$$\overline{u_i h_j'} = R \epsilon_{ijl} \xi_l \tag{6.7}$$

where $\xi_i = x_1' - x_1$, $r^2 = \xi_i \xi_i$, Q, H and R are scalar functions of r and t only. $\delta_{ij} = 0$ if $i \neq j$; $\delta_{ij} = 1$, if $i = j$ ϵ_{ijl} is the usual alternating symbol. $Q' = \frac{\partial Q}{\partial r}$, $H' = \frac{\partial H}{\partial r}$

Similarly for triple correlations, we have

$$\overline{u_i u_j u_k'} = \frac{2}{r} T' \xi_i \xi_j \xi_k - (rT' + 3T)(\xi_i \delta_{jk} + \xi_j \delta_{ik}) + 2T \delta_{ij} \xi_k$$

$$\overline{h_i h_j u_k'} = \frac{2}{r} S' \xi_i \xi_j \xi_k - (rS' + 3S)(\xi_i \delta_{jk} + \xi_j \delta_{ik}) + 2S \delta_{ij} \xi_k$$

$$\overline{u_i u_j h_k'} = U(\xi_i \epsilon_{jkl} \xi_l + \xi_j \epsilon_{ikl} \xi_l)$$

$$\overline{h_i h_j h_k'} = V(\xi_i \epsilon_{jkl} \xi_l + \xi_j \epsilon_{ikl} \xi_l)$$

$$\overline{(h_i u_j - h_j u_i) h_k'} = P(\xi_i \delta_{jk} - \xi_j \delta_{ik}) \tag{6.8}$$

$$\overline{u_i (h_j' u_k' - h_k' u_j')} = (2W + rW')\epsilon_{ijk} - \frac{W'}{r} \xi_i \epsilon_{jkl} \xi_l$$

where T, S, U, V, P and W are scalars which are functions of r
and t in homogeneous and isotropic turbulence. Prime on these
scalars refers to partial differentiation with respect to r.

From the fundamental equation (6.4), the equations governing these scalar functions may be found. The equation governing
Q, S, and T is

$$\frac{\partial Q}{\partial t} = 2\left(r\frac{\partial}{\partial r} + 5\right)(T-S) + \omega\left(\frac{\partial^2}{\partial r^2} + \frac{4}{r}\frac{\partial}{\partial r}\right)Q \qquad (6.9)$$

This is a generalization of the von Karman - Howarth equation to
magnetohydrodynamics. Similar to ordinary hydrodynamics, equation
(6.9) gives an invariant

$$\int_0^\infty Q(r,t)\, r^4 dr = CONSTANT \qquad (6.10)$$

This is known as Loitsiansky invariant. The existence of such an
invariant shows that no transfer of energy from the velocity field
to the magnetic field takes place among the largest eddies.

The equation governing H and P is

$$\frac{\partial H}{\partial t} = 2P + 2\omega_H\left(\frac{\partial^2}{\partial r^2} + \frac{4}{r}\frac{\partial}{\partial r}\right)H \qquad (6.11)$$

and the equation governing R, U, V and W is

$$\frac{\partial R}{\partial t} = \left(r\frac{\partial}{\partial r} + 5\right)(U-V) + \left(\frac{\partial^2}{\partial r^2} + \frac{4}{r}\frac{\partial}{\partial r}\right)\left[(\omega_H + \omega)R - W\right] \qquad (6.12)$$

From equations (6.11) and (6.12), we may obtain the invariants of
the Loitsiansky type for H and R respectively.

The rate of dissipation of energy may be obtained from (6.9)
and (6.10) by taking the limit of r→0. We have then

$$\frac{1}{2}\frac{d}{dt}\overline{|\vec{g}^2|} = -7.5\ \overline{h_1^2\frac{du_1}{dx_1}} - 60\ \omega\, Q_2$$

$$\frac{1}{2}\frac{d}{dt}\overline{|\vec{h}^2|} = +7.5\ \overline{h_1^2\frac{du_1}{dx_1}} - 60\ \omega_H H_2 \tag{6.13}$$

or

$$\frac{1}{2}\frac{d}{dt}\left(\overline{|\vec{g}^2|} + \overline{|\vec{h}^2|}\right) = -60\left(\omega\, Q_2 + \omega_H\, H_2\right) \tag{6.14}$$

where Q_2 and H_2 define the curvatures of the curves of longitudinal correlations $\overline{u_1 u_1'}$ and $\overline{h_1 h_1'}$ at $r = 0$ respectively. They represent the smallest eddies in the turbulent field. Equation (6.14) shows that the rate of dissipation is due to viscous dissipation and production of Joule heat by electrical conductivity.

7. Magnetohydrodynamic experiments.

There are very few experimental investigations of magnetohydrodynamic phenomena because it is necessary to have a strong magnetic field within a large volume in order to observe any appreciable magnetohydrodynamic effects. It is rather difficult to achieve in a laboratory.

The first magnetohydrodynamic experiment was done by Hartmann[6,7] and Lazarus[7]. They investigated the flow of mercury in a pipe perpendicular to a magnetic field. They found a qualitative agreement between theory ($\S$ 4) and experiments for laminar flow and discovered that the magnetic field has an influence on the transition

between laminar and turbulent flow.

Lehnert[3] recently reexamined Hartmann's data and found that the critical velocity for transition was proportional to the magnetic field strength. Lehnert also measured the torque transmitted by mercury contained between two non-conducting cylinders rotating in a magnetic field parallel to the axis. For laminar flow, the effect due to magnetic field is small. For non-laminar flow with the outer cylinder at rest and inner cylinder rotating, the torque was found to decrease when the magnetic field was applying. Lehnert planned to use molten-sodium instead of mercury in his new experiment because of higher conductivity and lower density than mercury. For instance the value of $R_H R_V^2$ for molten sodium is about 35 times that of mercury in the same physical conditions. However, there are many experimental difficulties to use molten sodium.

Lehnert[3] also showed a very interesting simple experiment to demonstrate the magnetohydrodynamic phenomenon. Consider a cylindrical glass vessel containing mercury. Without magnetic field, if one moves the mercury with a peg, the surface is agitated and complicated wave pattern occurs, particularly when one hits the vessel, surface waves occur just like water surface waves.

If the vessel is placed in a strong magnetic field perpendicular to the surface of the liquid, the surface waves disappear. The liquid no longer behaves like water but like thick syrup. If a peg is being moved in the liquid, the surface shows some

large whirls all with an axis of rotation parallel with the magnetic field lines.

Part II. MAGNETOGASDYNAMICS

8. Important parameters in magnetogasdynamics.

At very high temperature, the gas may be ionized, and the interaction of electromagnetic forces and the gasdynamic force may not be negligible. Furthermore, under such conditions, the compressibility effects should be considered. In this case our fundamental equations are (2.1) and (2.9). This system of equations has not been investigated yet. Only the simple one-dimensional waves of infinitesimal amplitude, Alfven's wave ($\S$ 9) and shock waves ($\S$ 10) have been discussed under further simplified assumptions. We shall briefly review these results later. First we should like to bring out the important para-meters in magnetogasdynamics. We shall restrict ourselves to the adiabatic case ($\frac{\partial Q}{\partial t} = 0$) with negligible radiation lost ($q_R = 0$). Furthermore, we shall consider the case where the velocity of the flow $\vec{q}$ is much smaller than the velocity of light C so that the relativistic effects may be neglected and that the energy in the electric field is negligible in comparison with that in the magnetic field.

Under these magnetogasdynamic approximations, the fundamental equations (2.1) to (2.9) may be reduced to the following non-dimensional form:

$$\rho^*\left[\frac{\partial \vec{q}^*}{\partial t^*} + (\vec{q}^* \cdot \nabla^*)\vec{q}^*\right] - R_H\left(\vec{H}^* \cdot \nabla^*\right)\vec{H}^* = -\nabla^*\left(p^* + \frac{R_H}{2}\vec{H}^{*2}\right) + \frac{1}{R_e}\nabla^* \cdot \mathcal{C}^{M*}$$

(8.1)

$$\frac{\partial \vec{H^*}}{\partial t^*} + (\vec{q^*} \cdot \nabla^*) \vec{H^*} - (\vec{H^*} \cdot \nabla^*) \vec{q^*} + \vec{H^*}(\nabla^* \cdot \vec{q^*}) = \frac{1}{R_V} \nabla^{*2} \vec{H^*}$$

(8.2)

$$\frac{\partial \rho^*}{\partial t^*} + \nabla^* \cdot (\rho^* \vec{q^*}) = 0$$

(8.3)

$$\gamma M^2 p^* = \rho^* T^*$$

(8.4)

$$\rho^* \frac{DT^*}{Dt^*} + \frac{\gamma-1}{2} M^2 R_H \left[\nabla^* \cdot (\vec{H^{*2}} \vec{q^*}) \right] = (\gamma-1) M^2 \frac{Dp^*}{Dt^*}$$

$$+ (\gamma-1) M^2 \left(\frac{1}{R_e} \tau_{ij}^{m*} + R_H \tau_{ij}^{H*} \right) \frac{\partial u_i^*}{\partial x_j^*} + \frac{1}{P_r R_e} \frac{\partial}{\partial x_j^*} \left(K^* \frac{\partial T^*}{\partial x_j^*} \right) \quad (8.5)$$

where $\rho^* = \frac{\rho}{\rho_o}$; $T^* = \frac{T}{T_o}$, $\tau_{ij}^{m*} = \left(\mu_o \frac{U}{L} \right)^{-1} \tau_{ij}^{m}$

$\tau_{ij}^{H*} = \frac{1}{\mu_e H_o^2} \tau_{ij}^{H}$, $\tau_{ij}^{H} = \mu_e H_i H_j - \frac{1}{2} \mu_e H_k H_k \delta_{ij}$

$\mu^* = \frac{\mu}{\mu_o}$, $K^* = \frac{K}{K_o}$, $\gamma = \frac{c_p}{c_v}$, $\frac{D}{Dt^*} = \frac{\partial}{\partial t^*} + u_i^* \frac{\partial}{\partial x_i^*}$

(8.6)

Mach number $= M = \frac{U}{\gamma M T_e}$, $P_r = \frac{\mu c_p}{K_o} =$ Prandtl number, and the
other non-dimensional quantities are same as those defined in
(3.4).

From these equations we see that as far as the magnetic
forces are concerned, the important parameters are still R_H and
R_V. The other new parameters are those well known parameters in
gas dynamics, i.e. P_r, M and γ.

9. Magnetogasdynamic waves - Alfven's wave.

The general solution of magnetogasdynamics is very difficult. In order to bring out some essential features of magneto-gasdynamics, we may investigate some very simple cases. One of these simple cases is one dimensional wave motion of small amplitude. Alfven was the first one who found such wave in magnetohydrodynamics[18]. If there is a homogeneous magnetic field H_0 in an incompressible and inviscid fluid of density ρ_0 and infinite conductivity $\sigma = \infty$, the disturbance in this liquid will propagate as a wave in the direction of H_0 with a speed of

$$V_H = \sqrt{\frac{\mu_e}{\rho_0}} \cdot H_0 \tag{9.1}$$

This wave is known as Alfven's wave.

Alfven's analysis has been extended to the case of compressible fluid by several authors[19-21]. The following analysis was due to van de Hulst[21].

Strictly speaking, to investigate the one dimensional wave motion in magnetogasdynamics, we should use the system of equation (8.1) to (8.5). But Van de Hulst made the following simplified assumptions:

(i) The energy equation is replaced by a simple adiabatic relation

$$p' = -C \operatorname{div} \vec{w} \tag{9.2}$$

where p' is the perturbed pressure from that of the gas at rest p_0 , $\vec{w}$ is the material displacement from the position at rest and the fluid velocity $\vec{q}$ is

$$\vec{q} = \frac{\partial \vec{w}}{\partial t} \tag{9.3}$$

$C = P_0 a_0^2$ = compressibility factor, a_0 is the velocity of sound of the gas at rest.

(ii) The coefficient of viscosity is assumed to be constant.

(iii) All the quantities are assumed to be function of one space coordinate Z and time t only. The perturbed quantities are assumed to be small so that the equations for the perturbed quantities may be linearized.

(iv) A homogeneous magnetic field H is applied. We wish to investigate wave travels in the Z-direction. The x-axis is chosen perpendicular to H so that the components of H are $(0, H_y, H_z)$.

If we substitute the assumptions into equations (2.1), (2.2), (2.3), (2.6), (9.2) and (9.3), simplify them under magnetogas-dynamic approximations and linearized, we have two independent sets of linear equations for the perturbed quantities; one set contains the perturbed quantities, h_y, E_x, J_x, q_y, q_z, p' and W_z, and the other set contains h_x, E_y, J_y, q_x, J_z and E_z.

We are looking for periodic solutions, in which all perturbed quantities are proportional to

$$e^{i\omega t - ikz} \tag{9.4}$$

where ω has a given real value. Only for certain eigenvalues of k, such solution exists. These eigenvalues of k give the different modes of wave propagation and they are determined from the determinantal equations of the two sets of linear equations

for the perturbed quantities. These two determinantal equations after simplifications are as follows

$$(1+af\chi_m)(x-f'\chi_s)(x-\mu_e\chi_e)-\mu_e\chi_m f(x-f\chi_s)-g\chi_s f(x-\mu_e\chi_e)=0 \quad (9.5)$$

$$x-(\mu_e-ax+ax_e\mu_e)\left[f\chi_m(1+ax_e)+(g+1)\chi_e\right]=0 \quad (9.6)$$

where

$$X=\frac{k^2}{\omega^2} \quad , \quad X_e=\epsilon \quad , \quad X_s=\frac{1}{a_o^2} \quad , \quad X_m=\frac{\rho_o}{\mu_e H_z^2}=\frac{1}{V_H^2}$$

$$g=\frac{H_y}{H_z} \quad , \quad a=\frac{i\omega}{\sigma} \quad , \quad b=\frac{i\mu\omega}{\rho_o} \quad , \quad f=1-bx \quad (9.7)$$

$$f'=1-\frac{4}{3}bx \quad , \quad i=\sqrt{-1}$$

Since f and f' are linear functions of x, equation (9.5) has three roots of x and (9.6) has two. These roots are the eigenvalues we are looking for. We shall not review all the special cases for x here but a few interesting ones.

(a) No external magnetic field,

The roots of (9.5) are

$$x = x_1 = x_e \longrightarrow \text{a damped electromagnetic wave} \quad (9.8a)$$

$$x = x_2 = \frac{1}{b} \longrightarrow \text{a viscous wave} \quad (9.8b)$$

$$x = x_s \left(1 - \frac{3}{4} bx_s\right) \longrightarrow \text{a damped sound wave} \quad (9.8c)$$

There is no coupling between these three fundamental modes. The roots of (9.b) are those of (9.8a) and (9.8b). The difference of modes with the same values of x is a difference of the plane of polarization.

(b) Propagation in the direction of the external field
g = 0. The sound waves are not affected. The other modes are
coupled and follow from the equation

$$(x - \mu_e x_e)(1 + a f x_m) - \mu_e x_m f = 0 \tag{9.9}$$

(c) Propagation perpendicular to the external field.
$\frac{1}{x_m} = 0$ but $g/x_m \neq 0$. The viscosity wave of (9.5) and the electro-
magnetic wave of (9.6) are not affected. The viscosity wave of
(9.6) is slightly affected. The two other important modes of
(9.5) are coupled and satisfy the equation

$$(x - f x_s)(a x - a \mu_e x_e - \mu_e) - \frac{g}{x_m} x_s (x - \mu_e x_e) = 0 \tag{9.10}$$

(d) Undamped waves. If the conductivity is zero, we have the
ordinary undamped light and sound waves.

If the damping terms are omitted, which correspond to infi-
nite conductivity ($\sigma = \infty$) and zero viscosity ($\mu = 0$), the
equations (9.5) and (9.6) reduce respectively to the following
simple form

$$1 - \frac{x_m \mu_e}{x - \mu_e x_e} - \frac{g x_s}{x - x_s} = 0 \tag{9.11}$$

and

$$x = \mu[x_m + (g + 1) x_e] \tag{9.12}$$

The two solutions of (9.11) depend on the relative magnitude of
sound velocity of the medium a_0 and the velocity of the magneto-
hydrodynamic wave V_H. There is a slow mode with velocity smaller

than sound velocity and a fast mode with velocity larger than sound velocity. If the magnetic field increases, the slow mode changes from the solution $x = x_m$ to the solution $x = (g + 1) x_s$. This mode represents a regular Alfven's wave with velocity V_H as long as V_H is much smaller than a_o. It changes to a kind of retarded sound wave for larger values of V_H. The fast mode starts out as an ordinary sound wave but changes to a modified Alfven's wave as soon as V_H is larger than a_o.

(e) Slightly damped wave. If both "a" and "b" are small, some first order effects due to damping may be found from (9.5) and (9.6). We will not discuss them here.

10. Shock waves.

In the last section, we consider the propagation of waves of infinitesimal amplitude in a compressible fluid. The study of waves of finite amplitude in magnetogasdynamics has not been carefully carried out. Only the analysis of analogues of the Rankine-Hugoniat equations for shock waves in an infinitely conducting fluid has been made by de Hoffman and Teller[22]. They discussed both relativistic and non-relativistic shock waves. But we shall restrict ourselves to the non-relativistic cases which have been systematically interpreted by Helfer[23].

One of the main assumptions in de Hoffman-Teller analysis is that the conductivity of the fluid is infinite. Even though the conductivity of the fluid is actually finite because of the

large dimension of interstellar clouds in which such an analysis is applied and the large conductivities of stellar materials, such an assumption is a reasonable one. Under this assumption the lines of magnetic force move with the fluid [of 4(c)]. Another assumption is that the fluid is assumed to be an ideal gas of a constant value of ratio of specific heats γ.

For ordinary shock relations, if we know the conditions in front of the shock, the conditions behind the shock depend on one parameter, the strength of the shock, e.g. the pressure ratio across the shock. In the case of magnetogasdynamics, the shock relations depend on three parameters:

(i) Strength of the shock. We may use the pressure ratio $\Upsilon = p_2/p_1$ as a measure of the strength of shock.

(ii) Magnetic field strength. We may use the ratio of magnetic energy per unit mass to internal energy $Q(\gamma - 1)$ as a measure of the importance of the magnetic field strength, where

$$Q = \frac{1}{\gamma - 1} \frac{\mu_e H_1^2/\rho_1}{p_1/(\gamma-1)\rho_1} \tag{10.1}$$

(iii) The direction of the magnetic field. We may use the angle between magnetic lines of force in front of the shock and the direction of propagation of shock θ or

$$s_1 = \tan \theta_1 \tag{10.2}$$

as a measure of the direction of magnetic field. The subscript 1 refers to values in front of shock and subscript 2 refers to values behind the shock,

According to the parameter s_1, we have the following three cases:

(a) Parallel shock $s_1 = 0$

In this case, the magnetic field has no influence upon the gas dynamic phenomena. Classical Rankine-Hugoniot relations for shock hold in this case.

(b) Perpendicular shock $s_1 = \infty$.

In the coordinate system of stationary shock with velocity q_1 in front of the shock and perpendicular to the shock front, we have the following shock relations in magnetogasdynamics.

$$\rho_1 q_1 = \rho_2 q_2 \tag{10.3}$$

$$\rho_1 q_1^2 + p^{*2} = \rho_2 q_2^2 + p_2^{*2} \tag{10.4}$$

$$\rho_1 q_1 I_1^{*} + q_1 p_1^{*} + \tfrac{1}{2}\rho_1 q_1^3 = \rho_2 q_2 I_2^{*} + q_2 p_2^{*} + \tfrac{1}{2}\rho_2 q_2^3 \tag{10.5}$$

$$p_1^{*} = p_1 + \frac{\mu_e H_1^2}{2} \quad , \qquad I_1^{*} = I_1 + \frac{\mu_e H_e^2}{2\rho_1} \tag{10.6}$$

$$H_1 q_1 = H_2 q_2 \tag{10.7}$$

Equations (10.3) and (10.5) have exactly the same forms as in the case of ordinary shock if one takes into account the magnetic pressure and energy by (10.6) and the coupling of magnetic and

velocity field by (10.7) which is a consequence of infinite conductivity. The magnetic field is compressed by the shock exactly to the same extent as the fluid

$$\frac{H_2}{H_1} = \frac{\rho_2}{\rho_1} = \eta \qquad (10.8)$$

The equation for η in terms of Y and Q is obtained from (10.3) to (10.7) as follows:

$$Q(\eta-1)^3 + \left[\frac{2\gamma}{\gamma-1} + (Y-1)\right](\eta-1) - \frac{2}{\gamma-1}(Y-1) = 0 \qquad (10.9)$$

When $Q = 0$, it reduces to the Rankine-Hugoniot relation. The presence of the magnetic field is seen to cause a decrease in compression η . The Mach numbers q_1/a_1 and q_2/a_2 are increased due to the presence of the magnetic field for the same reason.

(c) Oblique shocks

The oblique shock relations of this case has been worked out by Helfer. Numerical results for Q vs Y-1 at various values of s_1 are given by Helfer in reference 24. It is interesting to note that for very weak shock, $Y \simeq 1$, the shock relations tend to be the value of wave of infinitesimal amplitude discussed in § 9. Another interesting result is that for weak fields, the magnetic field is always amplified by the passage of shock fronts. This is important to explain some of the astrophysical phenomena[23].

References

1. Alfven, H.: Electromagnetic phenomena in the motion of gaseous masses of cosmic dimensions. Problems of cosmic Aerodynamics 1951, pp. 41-44. Published by Central Air Document Office, Dayton, Ohio.

2. Alfven, H.: Cosmical Electrodynamics, Oxford Press 1950, Chap. IV, pp. 76-97.

3. Lehnert, Bo.: On the behavior of an electrically conducting liquid in a magnetic field Arkiv. för Fysik Bd 5, No. 5, pp. 69-90, 1952.

4. Lundquist, S.: Studies in Magnetohydrodynamics Arkiv för Fysik, Bd. 5, No. 15, pp. 297-347, 1952.

5. Stratton, J.A.: Electromagnetic Theory, McGraw-Hill Book Company, Inc., N. Y., 1941.

6. Hartmann, J.: Hg-dynamics I, Kgl. Danke Vidensk, Selskab, Math-fys. Medd. XV:6, 1937.

7. Hartmann, J. and Lazarus, F.: Hg-dynamic II. ib. XV:7, 1937.

8. Pai, S. I.: Laminar flow of an electrically conducting incompressible fluid in a circular pipe. Jour. Appl. Phys., vol. 25, No. 9, pp. 1205-1207, Sept. 1954.

9. Shercliff, J.A.: Steady motion of conducting fluids in pipes under transverse magnetic fields, Proc. Cambridge, Phil. Soc., Vol. 49, pt. 1, pp. 136-144, 1953.

10. Chandrasekhar, S.: The stability of viscous flow between rotating cylinders in the presence of a magnetic field, Proc. Roy. Soc., London, A216, pp. 293-309, 1953.

11. Chandrasekhar, S.: On the inhibition of convection by a magnetic field. Phil. Mag. (7) 43, pp. 501-522, 1952.

12. Chandrasekhar, S.: Examples of the instability of fluid motion in the presence of a magnetic field. Proc. of Sym. in Appl. Math., vol. V, pp. 19-27, 1954, McGraw-Hill Company, Inc., N. Y.

13. Michael, D. H.: Stability of plane parallel flows of electrically conducting fluids. Proc. Cambridge Phil. Society 49, pp. 166-168, 1953.

14. Stuart, J. T.: On the stability of viscous flow between parallel planes in the presence of a coplanar magnetic field. Proc. Roy. Soc., London, A-221, pp. 186-206, 1954.

15. Squire, H. B.: On the stability for three dimensional disturbance of viscous fluid flow between parallel walls. Proc. Roy. Soc. London A-142. pp. 621-628, 1933.

16. Batchelor, G. K.: On the spontaneous magnetic field in a conducting liquid in turbulent motion. Proc. Roy. Soc., London A-201, pp. 405-416, 1950.

17. Chandrasekhar, S.: The invariant theory of isotropic turbulence in magnetohydrodynamics. I. Proc. Roy. Soc., London A-204, pp. 435-449, 1951. II. Proc. Roy. Soc., London A-207, pp. 301-306, 1951.

18. Alfven, H.: On the existence of electromagnetic hydrodynamic waves. Arkiv. f. Math., Astr. ock Fysik 29b, No. 2, pp. 1-7, 1943.

19. Aströar, E.: Magnetohydrodynamic waves in a plasma nature, vol. 165, p. 1019, 1950.

20. Herlofson, N.: Magnetohydrodynamic waves in a compressible fluid conductor, Nature, Vol. 165, p. 1020, 1950.

21. van de Hulst, H. C.: Interstellar Polarization and magnetohydrodynamic waves. Problems of cosmical Aerodynamics, pp. 46-56, 1951, Central Air Documents Office.

22. de Hoffman, F. and Teller, E.: Magnetohydrodynamic Shock, Phys. Rev., vol. 80, No. 4, pp. 692-703, Nov. 15, 1950.

23. Helfer, H. L.: Magnetohydrodynamic Shock Waves, The Astrophysical Jour., Vol. 117, pp. 177-199, 1953.